Inverse Matrices Linear Algebra X. Du

- Only applies to square matrices.
- An inverse matrix A^{-1} is the matrix such that $AA^{-1} = A^{-1}A = I$.
- If A^{-1} exists, then A is said to be **invertible**.
  - A is invertible iff $\det(A) \neq 0$.
- A^{-1} is unique is unique to A.
  - The inverse of A^{-1} is A.
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(cA)^{-1} = \dfrac{1}{c}A^{-1}$
- For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \dfrac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- How to find A^{-1}:
  - Method 1: Perform Gaussian elimination on the associated augmented matrix $\begin{pmatrix} A & I \end{pmatrix}$. The reduced row-echelon form should be the matrix $\begin{pmatrix} I & A^{-1} \end{pmatrix}$
    - If this process does not yield $\begin{pmatrix} I & A^{-1} \end{pmatrix}$, then A is not invertible.
  - Method 2: $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$, where $\operatorname{adj}(A)$ is the **adjugate matrix**, the transpose of the matrix of cofactors (determinant with the row and column deleted).
- **Invertible Matrix Theorem**: The following statements are equivalent (i.e. any one implies another) for some $n \times n$ square matrix A.
  - A is invertible.
  - $\det(A) \neq 0$
  - A^T is invertible
  - The reduced row-echelon form of A is I.
  - A can be written as a finite product of elementary matrices.
  - $A\vec{x} = \vec{b}$ has a unique solution
  - $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$
  - The row and column vectors of A are linearly independent.
  - The row and column vectors of A form a basis of R^n
  - A linear transformation $T : R^n \to R^n$, $T(\vec{x}) = A\vec{x}$ is a bijection.
  - 0 is not an eigenvalue of A.
  - 0 is not a singular value of A.
  - $\operatorname{rank}(A) = n$
  - $\ker(A) = \{0\}$